A Brief Overview of Black Hole Thermodynamics

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ABSTRACT

This paper explores the thermodynamic properties of black holes, discussing the derivation and implications of Hawking temperature. Black holes, traditionally studied within the framework of general relativity, exhibit thermodynamic characteristics such as entropy and temperature, which raise intriguing questions about their microscopic and quantum nature. A key result, the Bekenstein-Hawking entropy, establishes that black hole entropy scales with the surface area of the event horizon, rather than its volume, pointing to a fundamental connection between entropy and geometric properties. The derivation of Hawking temperature demonstrates how quantum effects near the event horizon lead to black hole radiation and slow evaporation. This paper provides a focused discussion on these aspects, avoiding deeper exploration of advanced topics such as black hole microstates, holography, and quantum gravity.

Keywords: Event horizon, Black hole thermodynamics, Bekenstein-Hawking entropy, Hawking radiation, Holographic principle, AdS/CFT correspondence

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I. INTRODUCTION

Black holes are one the most mysterious objects known to exist in the universe; the nature of black holes themselves tests our theories of gravity and quantum mechanics. Classically, black holes are studied within the context of general relativity, where they are defined as regions of spacetime where gravitational fields are such that nothing (including matter, light and information) can escape from its boundary (event horizon). But a series of surprising discoveries over the last few decades has uncovered the thermodynamic properties of black holes, which requires explanations from other fields such as statistical mechanics and quantum field theory, forcing deep questions on the nature of entropy, information, and the fundamental structure of space-time.

One of the most intriguing ideas of black hole thermodynamics is the concept of black hole entropy. In 1973, theoretical physicist Jacob Bekenstein discovered that black holes have entropy[1], described by the Bekenstein-Hawking entropy formula. According to the formula, the entropy of a black hole is proportional to the surface of its event horizon, but not to its volume. Such area-scaling behaviour is unusual and counter-intuitive, because for conventional thermodynamic systems, entropy scales with volume, reflecting the number of microscopic configurations accessible in three-dimensional space. The relation between area and entropy suggests that black holes are fundamentally described by the two-dimensional degrees of freedom of the horizon. This raises questions of what the "microstates" of a black hole are.

Another discovery related to this concept occurred in 1974, where Stephen Hawking proved that black holes emit radiation due to quantum effects near the event horizon. This radiation, now called Hawking radiation, implies that black holes have a temperature and would gradually lose mass though "evaporation"[2]. He also proved that black holes follow the laws of thermodynamics: they have a temperature inversely proportional to their mass. This provided not only a deeper link between black holes and thermodynamics but brought up the question of the black hole information paradox: it remains unclear how the information could be conserved during the process of black hole evaporation.

The peculiar behaviour of black hole entropy has inspired broader theories, especially the holographic principle, which states that all information contained in a volume of space can be represented by data on the surface enclosing that volume. It proposes that spacetime could be holographic in nature, providing a new insight into the structure of the universe that challenges our understanding of dimensionality in physical theories. The holographic principle has heavily inspired the AdS/CFT correspondence, which conjectures that a gravitational theory residing in higher-dimensional space can be equivalently described by a conformal field theory on its boundary.

This paper explores black hole thermodynamics through the tools of statistical mechanics, focusing on entropy, Hawking radiation, and the holographic principle, with a view on how black holes bridge classical thermodynamics, statistical mechanics, general relativity and quantum field theory, highlighting the main open questions and implications of black hole physics.

II. LAWS OF BLACK HOLE THERMODYNAMICS

The laws of black hole thermodynamics are a set of principles that reveal certain similarities between black holes and conventional thermodynamic systems. Formulated by Bardeen, Carter, and Hawking, the laws of black hole thermodynamics appeared to be identical to the laws of thermodynamics if the quantities involved, such as mass, surface area, and surface gravity, respectively, are treated analogously to energy, entropy, and temperature. These laws are summarized below, and the similarities to the three thermodynamics laws are discussed.

A. Zeroth Law of Thermodynamics

The zeroth law of black hole thermodynamics states that the surface gravity κ of a black hole remains constant across its event horizon:

$$\kappa = \text{constant on the horizon} \tag{2.1}$$

In thermodynamics, the zeroth law states that if two systems are in thermal equilibrium with a third, they are in equilibrium with each other. This defines temperature as a constant quantity in a system that exhibits thermal equilibrium. In the context of black holes, the zeroth law (of black hole thermodynamics) implies that surface gravity (analogous to temperature) is uniform across the event horizon of a stationary (non-rotating, unchanging) black hole. This constancy is essential for defining a black hole's "temperature" and serves as a foundation for the subsequent laws, particularly in describing black holes as thermodynamic entities.

This principle underpins the idea that a black hole's horizon behaves similarly to the boundary of a thermodynamic system, in which temperature remains the same throughout an isolated, equilibrium state. The zeroth law's role here is foundational, enabling the analogy between surface gravity and temperature and thus allowing black holes to be studied as systems with thermodynamic properties.

B. First Law of Thermodynamics

The first law of black hole thermodynamics establishes a relation between a black hole's mass, area, and angular momentum, analogous to the first law of thermodynamics, which describes the conservation of energy. For black holes, this law is expressed as:

$$dM = \frac{\kappa}{8\pi} dA + \vec{\Omega} \cdot d\vec{L} + \Phi dQ \tag{2.2}$$

where:

- M is the mass of the black hole, analogous to internal energy,
- κ is the surface gravity of the black hole, analogous to temperature,
- A is the surface area of the event horizon, analogous to entropy,
- $\vec{\Omega}$ and $\vec{L}$ represent the angular velocity and angular momentum, respectively,
- Φ and Q represent the electrostatic potential and electric charge, respectively.

Changes in the black hole's mass M can be attributed to changes in its horizon area, angular momentum, or charge. The surface gravity κ plays a role similar to temperature, dictating how changes in the black hole's horizon area (related to entropy) affect its energy.

C. Second Law of Thermodynamics

The second law of black hole thermodynamics, often called the area theorem, states that the total surface area A of a black hole's event horizon cannot decrease over time:

$$dA \geq 0 \tag{2.3}$$

This is strikingly similar to the second law of thermodynamics, which states that entropy never decreases in an isolated system. In black hole thermodynamics, the horizon area A is considered to be a type of entropy, which increases as the black hole absorbs matter. Proposed by Stephen Hawking, this principle means that black holes evolve toward states of higher entropy, identical to thermodynamic systems. The second law is not strictly true, as Hawking later predicted that black holes release energy in the form of radiation, which in turn causes the black hole to slowly lose mass. In such cases, the black hole area decreases over time, which contradicts with the second law of black hole thermodynamics. This paradox forms the basis for many ongoing research into the quantum mechanics of black holes. The Generalised Second Law of Thermodynamics was later introduced to maintain the validity of the second law of thermodynamics.

D. Third Law of Thermodynamics

The third law of black hole thermodynamics states that it is impossible to reduce a black hole's surface gravity κ to zero through any physical process. In classical thermodynamics, the third law implies that reaching absolute zero temperature is impossible. Here, it suggests that surface gravity, which acts as an analogue for temperature in black hole thermodynamics, cannot be eliminated entirely.

This law is particularly relevant when considering extremal black holes (black holes with the lowest possible temperature and maximum charge or angular momentum). Extremal black holes have surface gravities close to zero, yet the third law suggests that a true zero surface gravity state is unattainable. This implies a theoretical limit on cooling a black hole and underscores a deeper, possibly quantum-based, nature of black hole states at extreme conditions.

III. BEKENSTEIN-HAWKING ENTROPY AND HAWKING TEMPERATURE

Black holes have entropy, which are described by the Bekenstein-Hawking entropy formula, named after the theoretical physicists Jacob Bekenstein and Stephen Hawking. The formula can be derived using classical arguments[1]. Black holes also have temperature, and it emits radiation through a process known as Hawking radiation. This section includes both the classical and quantum derivation of the B-H entropy and Hawking radiation based on results from classical thermodynamics, general relativity, and quantum field theory.

A. Classical Derivation

Rotating charged black holes are described by the Kerr-Newman metric. If a black hole has the mass M, angular momentum $\vec{L}$ and electric charge Q, the radius of the inner (r_-) and outer (r_+) horizons are given by:

$$r_\pm = \frac{GM}{c^2} \pm \sqrt{\left(\frac{GM}{c^2}\right)^2 - a^2 - \frac{Q^2 G}{4\pi\varepsilon_0 c^4}} \tag{3.1}$$

where

$$\vec{a} = \vec{L}/Mc \tag{3.2}$$

The surface area and surface gravity are given as follows:

$$A = 4\pi(r_+^2 + a^2) \tag{3.3}$$

$$\kappa = \frac{2\pi c^2 (r_+ - r_-)}{A} \tag{3.4}$$

These are results of general relativity, and the derivations of these equations will not be discussed in this article.

Rewriting equation (3.3) in differential form, we obtain the following expression.

$$dA = 8\pi \left[r_+ dr_+ + \frac{\vec{L}M^{-2}}{c^2} d\vec{L} - \frac{L^2 M^{-3}}{c^2} dM \right] \tag{3.5}$$

The outer horizon, r_+ is given by equation (3.1). In differential form:

$$\begin{aligned} dr_+ =& \frac{Gc^{-2} r_+ + L^2 M^{-3} c^{-2}}{\sqrt{\left(\frac{GM}{c^2}\right)^2 - \left(\frac{L}{Mc}\right)^2 - \frac{Q^2 G}{4\pi\varepsilon_0 c^4}}} dM \\ &- \frac{\vec{L}M^{-2}c^{-2}}{\sqrt{\left(\frac{GM}{c^2}\right)^2 - \left(\frac{L}{Mc}\right)^2 - \frac{Q^2 G}{4\pi\varepsilon_0 c^4}}} \cdot d\vec{L} \\ &- \frac{\frac{1}{4\pi\varepsilon_0} QGc^{-4}}{\sqrt{\left(\frac{GM}{c^2}\right)^2 - \left(\frac{L}{Mc}\right)^2 - \frac{Q^2 G}{4\pi\varepsilon_0 c^4}}} dQ \end{aligned} \tag{3.6}$$

Substituting (3.6) back into dA:

$$\begin{aligned} \frac{dA}{8\pi} =& \left(\frac{r_+(Gc^{-2} r_+ + L^2 M^{-3} c^{-2})}{\sqrt{\left(\frac{GM}{c^2}\right)^2 - \left(\frac{L}{Mc}\right)^2 - \frac{Q^2 G}{4\pi\varepsilon_0 c^4}}} - \frac{L^2}{M^3 c^2} \right) dM \\ &- \left(\frac{r_+ \vec{L} M^{-2} c^{-2}}{\sqrt{\left(\frac{GM}{c^2}\right)^2 - \left(\frac{L}{Mc}\right)^2 - \frac{Q^2 G}{4\pi\varepsilon_0 c^4}}} - \frac{\vec{L}}{M^2 c^2} \right) \cdot d\vec{L} \\ &- \frac{\frac{1}{4\pi\varepsilon_0} r_+ QGc^{-4}}{\sqrt{\left(\frac{GM}{c^2}\right)^2 - \left(\frac{L}{Mc}\right)^2 - \frac{Q^2 G}{4\pi\varepsilon_0 c^4}}} dQ \\ &\vdots \end{aligned}$$

$$dA = \frac{8\pi G}{\kappa} \left[dM - \frac{4\pi\vec{a}}{Ac} \cdot d\vec{L} - \frac{r_+ Q}{\varepsilon_0 A c^2} dQ \right] \tag{3.7}$$

Rearranging this equation to get an expression for dM:

$$\begin{aligned} dM &= \frac{\kappa}{8\pi G} dA + \frac{4\pi\vec{a}}{Ac} \cdot d\vec{L} + \frac{r_+ Q}{\varepsilon_0 A c^2} dQ \\ &= \frac{\kappa}{8\pi G} dA + \vec{\Omega} \cdot d\vec{L} + \Phi dQ \end{aligned} \tag{3.8}$$

where

- $\vec{\Omega} = \frac{4\pi\vec{a}}{Ac}$ is the angular velocity of the black hole.
- $\Phi = \frac{r_+ Q}{\varepsilon_0 A c^2}$ is the electrostatic potential of the black hole.

We have just derived the first law of black hole thermodynamics. Comparing equation (3.8) with the equation of internal energy of a thermodynamic system:

$$dE = TdS - PdV$$

It is clear that the latter two terms in (3.8), $\vec{\Omega} \cdot d\vec{L} + \Phi dQ$, represents the work done when increasing the black hole's charge and angular momentum. Therefore $\vec{\Omega} \cdot d\vec{L} + \Phi dQ$ is the analogue of $-PdV$, strongly suggesting that $\frac{\kappa}{8\pi G} dA$ is the black hole analogue of TdS, (differing with a constant).

Now we have to try and understand the specific relation $S = f(A)$ between black hole entropy S and the surface area A. It would be reasonable to assume that $f(A)$ is an increasing monotonic function, since entropy never decreases.

Consider functions $f(A)$ such that $f(A) = o(A)$. If the entropy of a black hole were related to its area in this manner, it would lead to inconsistencies. Suppose we have two black holes with areas A_1, A_2 and entropies S_1, S_2, and they merge to form a larger black hole. The total entropy of the system cannot decrease, implying $S \geq S_1 + S_2$. Since $S = f(A) = o(A)$, the resulting area A would **always exceed** the sum of the initial areas, $A > A_1 + A_2$. For two massless, uncharged (Schwarzschild) black holes, we could use Equation 2.2 to see that the final mass would also have to exceed the combined mass of the original black holes. Yet, this would contradict the expectation that energy is radiated as gravitational waves during the merge, making an overall increase in mass impossible. Thus, an entropy-area relation of $S = o(A)$ would be physically inconsistent.

The next simplest assumption $S = CA$, where C is a constant, implies a black hole's entropy is directly proportional to its surface area. This is exactly what Bekenstein proposed: the area A and surface gravity κ *are* the entropy and temperature of the black hole (with the inclusion of a constant factor.)

If C is indeed a constant for a black hole, it's sensible to assume that the constant has the same value for all black holes. Furthermore, since the constant is not a dimensionless one, we now make a bold assumption that it's based on known fundamental physical constants. Given the context of black holes, it's plausible that C involves the speed of light c and the gravitational constant G. There's no need to consider ε_0 as no charge is involved in Schwarzschild black holes. Note that SI base units of entropy contains the unit of temperature K. Given the involvement of thermodynamics, we also assume that the Boltzmann constant k_B is involved. And finally, C may also be related to the Planck's constant $\hbar$. Although it may seem counter-intuitive for the Planck constant to appear in a classical derivation, it should be noted that Planck constant also appears in many results of classical statistical mechanics. Thermodynamics seems to "predict" the existence of $\hbar$.

Combining all of these together:

$$S = \lambda k_B^\alpha c^\beta G^\gamma \hbar^\delta A \tag{3.9}$$

where λ is a dimensionless constant. Now we use the method of dimensional analysis to calculate the exact form of the equation. In SI base units:

- k_B has units of $kg\, m^2\, s^{-2}\, K^{-1}$.

- c has units of $m\,s^{-1}$.
- $\hbar$ has units of $kg\,m^2\,s^{-1}$.
- G has units of $kg^{-1}\,m^3\,s^{-2}$.
- S has units of $kg\,m^2\,s^{-2}\,K^{-1}$.
- A has units of m^2

Substituting these values into Equation (3.9), we determine the values of α, β, γ, and δ, leading to the following expression for entropy:

$$S = \boxed{\frac{\lambda k_B c^3}{\hbar G} A} \tag{3.10}$$

B. Quantum Derivation

The metric of a Schwarzchild black hole is as follows:

$$ds^2 = -\left(1 - \frac{2GM}{c^2 r}\right) c^2 dt^2 + \frac{dr^2}{1 - \frac{2GM}{c^2 r}} + r^2 d\Omega^2$$

where $d\Omega^2 = d\theta^2 + sin^2\theta\, d\phi^2$ is the metric on the two-sphere.

Consider an observer at a point just outside the horizon, i.e. $\rho \to 0^+$:

$$r = \frac{2GM}{c^2} + \frac{c^2\rho^2}{8GM} \tag{3.11}$$

Substituting this into the Schwarzchild metric, we obtain

$$\begin{aligned}
ds^2 &= -\frac{c^4\rho^2}{16G^2M^2 + c^4\rho^2} c^2 dt^2 \\
&\quad + \left(1 + \frac{c^4\rho^2}{16G^2M^2}\right) d\rho^2 \\
&\quad + \left(\frac{2GM}{c^2} + \frac{c^2\rho^2}{8GM}\right)^2 d\Omega^2 \\
&\approx -\frac{c^4\rho^2}{16G^2M^2}(d\,ct)^2 + d\rho^2 + \left(\frac{2GM}{c^2}\right) d\Omega^2 \\
&= -\left(\frac{\rho}{c^2}\right)^2 \left(\frac{c^4}{4GM}\right)^2 (d\,ct)^2 + d\rho^2 + \left(\frac{2GM}{c^2}\right) d\Omega^2
\end{aligned}$$

The frame of the observer is constantly accelerating to not fall into the black hole. The local acceleration is $\alpha = \frac{c^2}{\rho}$. Canadian theoretical physicist W. Unruh has shown that all uniformly accelerating frames will perceive a thermal bath, an effect in quantum field theory known as the Unruh effect[3]:

$$T(r) = \frac{\hbar\alpha}{2\pi c k_B} = \frac{\hbar c}{2\pi k_B \rho} \tag{3.12}$$

This is the temperature just outside the horizon, which is different to the temperature at any point r', due to gravitational redshift:

$$\begin{aligned}
T(r') &= \frac{\hbar c}{2\pi k_B \rho}\sqrt{\frac{1 - \frac{2GM}{c^2 r}}{1 - \frac{2GM}{c^2 r'}}} \\
&= \frac{\hbar c}{2\pi k_B \sqrt{\frac{8GMr}{c^2}\left(1 - \frac{2GM}{c^2 r'}\right)}}
\end{aligned}$$

The temperature at infinity is therefore:

$$\boxed{T(\infty) = \frac{\hbar c}{2\pi k_B \sqrt{\frac{8GM}{c^2}\frac{2GM}{c^2}}} = \frac{\hbar c^3}{8\pi G k_B M}} \tag{3.13}$$

This temperature is also known as *Hawking temperature*. Knowing the temperature, We can easily calculate the entropy of the black hole. Note that the heat transfer added to a black hole is all converted to an increase in its mass, i.e. $dQ = dMc^2$

$$dS = \frac{dQ}{T} = \frac{8\pi k_B GM}{\hbar c^3} dMc^2 = \frac{4\pi k_B G}{\hbar c} dM^2$$

For Schwarzchild black holes, $R = \frac{2GM}{c^2}$. Therefore,

$$\boxed{S = \frac{k_B c^3}{4\hbar G} A} \tag{3.14}$$

IV. HAWKING RADIATION AND THE LIFETIME OF BLACK HOLES

Hawking predicted that black holes gradually lose mass and can eventually evaporate through a process called Hawking radiation[2], which arises from quantum effects near the event horizon. Virtual particle pairs form near the horizon, and one particle may escape while the other falls into the black hole. To conserve energy, the escaping particle draws energy from the black hole, reducing its mass. This radiation causes the black hole to shrink, with the process accelerating as the black hole gets smaller.

The specific heat for this process is expressed as $C = \frac{1}{M}\left(\frac{dQ}{dT}\right)$ where M is the mass of the black hole, and dQ is the heat change (radiated away as Hawking radiation) by the black hole[4].

$$C = \frac{1}{M}\left(\frac{dQ}{dT}\right) = \frac{1}{M}\frac{dMc^2}{\frac{-\hbar c^3}{8\pi k_B GM^2}dM} = -\frac{8\pi k_B G}{\hbar c} M \tag{4.1}$$

Note that the specific heat is negative: $C < 0$. This implies that as the black hole's mass and size decrease, its temperature *increases*, and so does the rate of its energy loss. The rate of energy loss can be calculated by the Stefan-Boltzmann radiation law:

$$J = \frac{dU/dt}{A} = \sigma T^4 \tag{4.2}$$

where J is the total energy radiated by a unit surface area per unit time, A is the surface area, $U\left(= Mc^2\right)$ is the internal energy, and σ is the Stefan-Boltzmann constant, which has the exact value of $\frac{\pi^2 k_B^4}{60\hbar^3 c^2}$. It follows that

$$\begin{aligned}
\frac{dU}{dt} &= A\sigma T^4 = 4\pi R^2 \sigma T^4 \\
&= 4\pi\left(\frac{2GM}{c^2}\right)^2 \left(\frac{\pi^2 k_B^4}{60\hbar^3 c^2}\right)\left(\frac{\hbar c^3}{8\pi G k_B M}\right)^4 \\
\frac{dM}{dt} &= \frac{\hbar c^4}{15360\pi G^2 M^2} \\
dt &= \frac{5120\pi G^2}{\hbar c^4} dM^3
\end{aligned}$$

Integrating both sides, we obtain the lifetime of a black hole:

$$\boxed{t = \frac{5120\pi G^2}{\hbar c^4} M^3} \tag{4.3}$$

According to this formula, an average black hole that weighs around 10 solar masses has a lifetime of around 2×10^{70} years. For comparison, the age of the universe is only 13.8 billion years, nearly negligible compared the lifetime of a black hole.

There are other factors to consider here. For a black hole of the size mentioned above, its Hawking temperature would be around 6nK, a temperature much lower than the cosmic background radiation (CMBR), which has a current value of around 2.7K. Most black holes in the universe today are radiating so slowly that they are absorbing more energy from the cosmic microwave background radiation (CMBR) than they are emitting, effectively preventing them from beginning the process of evaporation.

V. DISCUSSION

Black hole evaporation through Hawking radiation is a slow process for most of a black hole's life, as the radiation is weak for large black holes with low surface gravity. However, as a black hole loses mass, the process accelerates, leading to a rapid increase in temperature and radiation output. In the final stages of evaporation, the black hole is expected to release a burst of high-energy gamma rays, potentially detectable as a significant astrophysical event. Despite theoretical predictions, no such gamma-ray bursts have been observed, possibly because the remaining black holes in the universe are still far from reaching this end stage. Hawking radiation appears to be a purely thermal phenomena and thus contains no information about the matter that formed the black hole. This violates unitarity, a key principle of quantum mechanics. Information seems to be permanently lost as the object passes through an event horizon, and this violation is known as the black hole information paradox.

The holographic principle is a remarkable idea inspired by the entropy-area relationship of black holes. It has been proposed that the dynamics within a region of spacetime can be fully described by data on its surface. Famously quoted by Susskind "The three-dimensional world of ordinary experience—the universe filled with galaxies, stars, planets, houses, boulders, and people—is a hologram, an image of reality coded on a distant two-dimensional surface." The holographic principle partly resolves the black hole information paradox. In the context of black holes, the holographic principle would imply that the information about the matter that fell into the black hole is stored on the event horizon and does not vanish when the black hole evaporates.

Anti-de Sitter spaces, characterised by a constant negative curvature, provide an ideal mathematical framework for studying gravity and field theories in lower-dimensional systems. These spaces are solutions to Einstein's equations with a negative cosmological constant, making them conceptually distinct from the universe's observed de Sitter-like expansion. AdS spaces feature boundary structures that are particularly well-suited for exploring holographic dualities and other theoretical constructs.

Conformal Field Theories describe quantum fields that are invariant under conformal transformations, which include scaling and rotations. These symmetries allow CFTs to describe critical phenomena in condensed matter systems and quantum systems at high energies. The AdS/CFT correspondence, proposed by Juan Maldacena, provides a duality between a gravitational theory in an AdS space and a CFT on its boundary. This correspondence connects quantum field theory and general relativity, offering insights into both the microscopic structure of spacetime and the behaviour of strongly interacting quantum systems. The duality has provided tools for exploring the thermodynamics of black holes, the behaviour of quark-gluon plasmas, and even condensed matter systems. Though its full implications remain to be uncovered, AdS/CFT stands as one of the most significant advances in theoretical physics, bridging disparate areas of study and suggesting pathways toward a unified framework for quantum gravity.

AdS/CFT provides a framework for understanding the Bekenstein-Hawking entropy of black holes. In this duality, black holes in AdS spaces correspond to thermal states in the boundary Conformal Field Theory (CFT). The entropy of these CFT states, computed using statistical mechanics, matches the geometric entropy of the AdS black hole. This offers a microscopic explanation for black hole entropy in terms of degrees of freedom in the dual theory, addressing one of the long-standing questions in black hole thermodynamics.

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